June 10 midterm exam
3:05-3:15pm today OCEP

Last time · differentiability ≈ linear approximation.
(" ⇒ continuous)
· $C^1 \Rightarrow$ differentiable. (proof: mean value theorem)
· Gradient & directional derivative

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

· f is differentiable at $\vec{a}$, $D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$

· ∇f : the direction f increases most rapidly
(at a rate $\|\nabla f\|$)
$-\nabla f$: " " decreases "

Total differential (midterm x)

Let $f: \Omega (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$ differentiable at $\vec{a} \in \Omega$.

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \varepsilon(\vec{x})$$

Denote $\Delta f = f(\vec{x}) - f(\vec{a})$

$\Delta x_i = x_i - a_i$

2nd order $\Rightarrow \|\vec{x}-\vec{a}\|^2$

$$\Delta f \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\,\Delta x_i$$

This approximation is good up to $\underbrace{\text{1st order}}_{f}$:

$$\lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0 \qquad \|\vec{x}-\vec{a}\|$$

This 1st order approximated change is denoted by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\,dx_i$$

and is called the total differential of f of $\vec{a}$.

Rmk In this class, the meaning of df, dx_i will not be explained further. In more advanced level, df, dx_i can be interpreted as linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.

eg $V(r, h) = \pi r^2 h$

V is $C^1 \Rightarrow V$ is differentiable

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$= 2\pi r h\, dr + \pi r^2 dh$$

Suppose we want to approximate change of V when (r, h) change from $(3, 12)$ to $(3+0.08,\ 12-0.3)$

Let $dr = \Delta r = 0.08$, $dh = \Delta h = -0.3$

Then $\Delta V \approx dV$

actual change of V

$$\begin{aligned} &= 2\pi r h\, dr + \pi r^2 dh \\ &= 2\pi \cdot 3 \cdot 12 \cdot 0.08 + \pi \cdot 3^2 \cdot (-0.3) \\ &= 3.06\pi \approx 9.61 \end{aligned}$$

<u>Properties of total differential</u>

$f, g : \Omega\ (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$ differentiable

c : a constant

① $d(f \pm g) = df \pm dg$, $d(cf) = c\,df$

② $d(fg) = g\,df + f\,dg$

③ $d(f/g) = \dfrac{g\,df - f\,dg}{g^2}$ (pf) easily follows from partial differentials.

Summary: Differentiating a real-valued function $f(\vec{x}) = f(x_1, \cdots x_n)$ at $\vec{a} \in \mathbb{R}^n$

- directional derivative : $D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$

$\cup$

- partial derivative : $\frac{\partial f}{\partial x_i}(\vec{a}) = f_{x_i}(\vec{a}) = D_{e_i} f(\vec{a})$

$$e_i = (0 - 1 - 0)$$

$\uparrow$ i-th

- Gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right)$
- total differential $df = \sum_1^n \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$
- higher derivatives $\frac{\partial^2 f}{\partial x_1 \partial x_2}$, $\frac{\partial^3 f}{\partial x_1^3}$ etc.

f is C^k if f and its all higher derivatives up to order k exist and are continuous.

linear approximation

- $L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$
- $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$

- f is differentiable at $\vec{a}$ if
$$\lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0$$

Relations ① C^0 (continuous) $\Leftarrow C^1 \Leftarrow C^2 \Leftarrow \cdots \Leftarrow C^\infty$

② f is C^1 at $\vec{a}$

$\Downarrow \quad \not\Uparrow$

f is differentiable at $\vec{a}$

$\not\nearrow \quad \swarrow \qquad\qquad \searrow \quad \not\nwarrow$

$D_{\vec{u}} f(\vec{a})$ exists for any unit vector $\vec{u} \in \mathbb{R}^n$ $\quad \not\Rightarrow \quad \not\Leftarrow \quad$ f is continuous at $\vec{a}$.

$\not\Uparrow \quad \Downarrow \qquad\qquad \not\swarrow \quad \not\nearrow$

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists for $i = 1, \cdots, n$

③ All the reverse of $\Rightarrow$ are false.

eg 1 $\quad C^k \Rightarrow C^{k-1}$ but $C^{k-1} \not\Rightarrow C^k$

Let $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

f is differentiable on $\mathbb{R}$ $f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

$$\lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

$f'(x)$ is not continuous at $x=0$.

In particular, differentiable $\not\Rightarrow C^1$

continuity $\not\Rightarrow C^1$
C^0

Similarly, $g(x) = x^{2k-2} f(x)$ is differentiable k-times, but $g^{(k)}(x)$ is not continuous at $x=0$.

$\therefore$ k-times differentiable $\not\Rightarrow C^k$

In particular $C^{k-1} \not\Rightarrow C^k$

multi-variable example: let $h(x_1, \cdots x_n) = g(x_1)$.

eg2 $$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

f is not continuous at $(0,0)$:

limit along $x=y^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

limit along $x=-y^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=-y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{-y^4}{y^4+y^4} = -\frac{1}{2}$$

$\therefore$ f is not continuous at $(0,0)$.

$D_{\vec{u}}f(0,0)$ exists for any unit vector $\vec{u} \in \mathbb{R}^n$.
$\vec{u} = (u_1, u_2)$

If $u_1 \neq 0$
$\vec{u} \neq e_2 = (0,1)$

$$D_{\vec{u}}f(0,0) = \lim_{t\to 0} \frac{f(t\vec{u}) - f(\vec{0})}{t}$$

$$= \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(\vec{0})}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^3u_1u_2^2}{t^2u_1^2+t^4u_2^4} - 0}{t}$$

$$= \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^3 u_1 u_2^2}{t^2 u_1^2 + t^4 u_2^4}$$

$$= \lim_{t\to 0} \frac{u_1 u_2^2}{u_1^2 + t u_2^4}$$

(if $u_1 \neq 0$)

$$= \frac{u_2^2}{u_1}$$

If $\vec{u} = (0,1)$

$$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\vec{u}) - f(\vec{0})}{t} = \lim_{t\to 0} \frac{f(0,t) - f(\vec{0})}{t}$$

$$= 0$$

<u>eg 3</u> $f(x,y) = |x+y|$

f is continuous on $\mathbb{R}^2$.

But $f_x(0,0)$, $f_y(0,0)$ does not exist.

<u>eg 4</u> $f(x,y) = \sqrt{|xy|}$

$f_x(0,0)$, $f_y(0,0)$ exist but

if $\vec{u} \neq e_1, e_2, -e_1, -e_2$, then

$D_{\vec{u}} f(0,0)$ does not exist.

$$f_x(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{0-0}{t} = 0.$$

Similarly, $f_y(0,0) = 0$.

$\vec{u} = (u_1, u_2)$

$$D_{\vec{u}} f(0,0) = \lim_{t \to 0} \frac{f(t\vec{u}) - f(\vec{0})}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{|t^2 u_1 u_2|} - 0}{t}$$

$$= \lim_{t \to 0} \frac{|t|}{t} \sqrt{|u_1 u_2|}$$

does not exist if

$\vec{u} \neq \pm\vec{e_1}, \pm\vec{e_2}$

$f: \mathbb{R}^2 \to \mathbb{R}$

Exercise $\vec{u}$ a unit vector in $\mathbb{R}^n$.

f_x, f_y continuous on Ω.

Show that $D_{\vec{u}} f$ is continuous on Ω.

[sol] Since f is C^1-function, [f is differentiable on Ω.]

Hence $\left[D_{\vec{u}} f(\vec{a}) = \nabla f \cdot \vec{u} \right]$

Let $\vec{u} = (u_1, u_2)$ $= (f_x, f_y) \cdot (u_1, u_2)$

$$= u_1 f_x + u_2 f_y$$

[Since f_x, f_y are continuous,] $u_1 f_x, u_2 f_y$ are continuous hence $u_1 f_x + u_2 f_y$ is continuous.

Exercise $g: \mathbb{R}^2 \to \mathbb{R}$ differentiable.

$f: \mathbb{R}^2 \to \mathbb{R}$ a real-valued function.

There exist $\delta > 0$ and a positive constant C

s.t. $|f(x,y) - g(x,y)| \leq C \cdot \sin^2(\|(x,y)\|)$

for all $(x,y) \in B_\delta(0,0)$

Is f differentiable at $(0,0)$?
If so, prove it, If not, provide a counter-example.

(proof) Yes.

$$L(x,y) = g(0,0) + g_x(0,0)x + g_y(0,0)y$$

$$\varepsilon(x,y) = f(x,y) - L(x,y)$$

$$\left(\text{Guess}) \lim_{(x,y)\to(0,0)} \frac{|\varepsilon(x,y)|}{\|(x,y)\|} = 0 \right)$$

$$\frac{|\varepsilon(x,y)|}{\|(x,y)\|} = \frac{|f(x,y) - L(x,y)|}{\|(x,y)\|}$$

$$= \frac{|f(x,y) - g(x,y) + g(x,y) - L(x,y)|}{\|(x,y)\|}$$

$$\leq \frac{|f(x,y) - g(x,y)|}{\|(x,y)\|} + \frac{|g(x,y) - L(x,y)|}{\|(x,y)\|}$$

for all $(x,y) \in B_\delta(0,0)$

Note that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|g(x,y) - L(x,y)|}{\|(x,y)\|} = 0$

because g is differentiable and $L(x,y)$

is the linear approximation of g.

$$\lim_{(x,y)\to(0,0)} \frac{C\cdot \sin^2(\|(x,y)\|)}{\|(x,y)\|} = \lim_{r\to 0} \frac{C\cdot \sin^2 r}{r}$$

(polar coordinate)

$$= \lim_{r\to 0} C\cdot \frac{\sin r}{r}\cdot \sin r$$

$$= 0$$

By the sandwich theorem,

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|(x,y)\|} = 0$$

$\therefore$ f is differentiable at $(0,0)$.

Exercise Evaluate $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2yz\cos(yz)}{x^4+y^2+z^2}$

(sol) $x'=x^2,\ y'=y,\ z'=z$

$$= \lim_{(x',y',z')\to(0,0,0)} \frac{x'y'z'\cos(y'z')}{x'^2+y'^2+z'^2}$$

$|x'|, |y'|, |z'| \leq \sqrt{x'^2+y'^2+z'^2}$

$|\cos(y'z')| \leq 1$

~~By sandwich thm.~~

$$\left| \frac{x'y'z' \cos(y'z')}{x'^2+y'^2+z'^2} \right| \leq \sqrt{x'^2+y'^2+z'^2}$$

$$\lim_{(x',y',z') \to (0,0,0)} \sqrt{x'^2+y'^2+z'^2} = 0.$$

By sandwich theorem,

$$\lim_{(x',y',z') \to (0,0,0)} f \ \text{''} = 0.$$